On Skew Primeness of Inner Functions

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Dedicated to Chandler Davis

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ABSTRACT

The notion of skew primeness introduced by Wolovich in the context of polynomial matrices is extended to the context of inner functions. Skew primeness is related to a geometric condition as well as to the solvability, over H^∞, of the Sylvester equation.

1. INTRODUCTION

One of the central motives in operator theory as well as in the polynomial model approach to linear algebra is the relation between factorization theory and invariant subspaces. Roughly stated, there is a bijective correspondence between invariant subspaces of certain operators and the factorization of their characteristic functions.

This leads to a very natural question: Given an invariant subspace of a linear transformation, when does it have a complementary invariant subspace? Moreover, given that a complementary invariant subspace exists, when is it unique?

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The problem addressed in this paper was raised first in Wolovich (1978) in the context of polynomial matrices. Khargonekar, Georgiou, and Ozgüler (1983) have studied the question of skew primeness in the context of polynomial models. Recently some of the results have been extended by Chang and Georgiou (1993) to the case of rational H^∞ functions. The connection with the Sylvester equation is pointed out in this work.

Our aim in this short paper is to give a discussion of the problem of skew primeness of matrix inner functions. No assumptions of rationality are made. The conditions for uniqueness are characterized. It is of interest to derive, in the rational case, computable algorithms for the characterization of skew primeness. The natural terms for such a characterization would be state space realizations and the use of the algebraic Riccati equation.

2. PRELIMINARIES

The context we will work in is that of Hardy spaces. Hoffmann (1962) is an excellent source for the necessary background on these spaces. For the problem at hand there is absolutely no difference if our fundamental domain is the unit disc or (say) the left half plane. So we will assume that our function spaces relate to the unit disc.

Inner functions are of great importance, due to the work of Beurling (1949) giving a characterization of the invariant subspaces of the shift operator. This work has been extended to the case of shifts of higher multiplicity by Halmos (1961) and Lax (1959).

For an inner function P the subspace PH^2 is the representation of a general invariant subspace. For modeling purposes its orthogonal complement, in H^2, is even more important. We will use the notation

$$H(P) = \{PH^2\}^\perp \tag{1}$$

for the orthogonal complement. Clearly $H(P)$ is an invariant subspace for the backward shift S^*. Thus the restriction $S^*|H(P)$ is a contractive operator. For notational convenience we will define the compression of the shift, $S_P : H(P) \to H(P)$, by

$$S_P f = P_{H(P)} zf \qquad \text{for} \quad f \in H(P). \tag{2}$$

An easy computation leads to

$$S_P^* = S^*|H(P). \tag{3}$$

Both S_P and its adjoint are model operators and play a prominent role in operator theory, in particular the Sz.-Nagy–Foias (1970) theory of contractions, as well as in realization theory for nonrational transfer functions, as in Fuhrmann (1981).

The important fact concerning model operators is the following result, going back at least to Helson (1964).

THEOREM 2.1. *Let P be an inner function. Then $\mathcal{V}$ is an S_P invariant subspace of $H(P)$ if and only if*

$$\mathcal{V} = P_1 H(P_2) \tag{4}$$

where $P = P_1 P_2$ is a factorization of the inner function P into the product of inner functions.

THEOREM 2.2 (CLT). *A Map $Z: H(R) \to H(\bar{R})$ intertwines S_R and $S_{\bar{R}}$, i.e. satisfies $ZS_R = S_{\bar{R}}Z$, if and only if there exist H^∞ functions A and $\bar{A}$ satisfying*

$$\bar{A}R = \bar{R}A \tag{5}$$

in terms of which Z is given by

$$Zf = P_{H(\bar{R})}\bar{A}f \qquad \text{for} \quad f \in H(R). \tag{6}$$

THEOREM 2.3 (Fuhrmann). *Let $Z: H(R) \to H(\bar{R})$ intertwine S_R and $S_{\bar{R}}$ and be given by (6). Then Z is invertible if and only if we have X, $\bar{X}$, Y, $\bar{Y} \in H^\infty$ satisfying the following Bezout equations:*

$$\begin{aligned} \bar{A}X + \bar{R}Y &= I, \\ \bar{X}A + \bar{Y}R &= I. \end{aligned} \tag{7}$$

Equations (6) and (7) can be embedded neatly in the matrix equation

$$\begin{pmatrix} \bar{Y} & \bar{X} \\ -\bar{A} & \bar{R} \end{pmatrix} \begin{pmatrix} R & -X \\ A & Y \end{pmatrix} = \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}$$

with $Q = \bar{X}Y - \bar{Y}X$. Multiplying on the right (left) by

$$\begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}$$

and redefining appropriately X, Y $(Y, \bar{X})$, we may assume without loss of generality that

$$\begin{pmatrix} \bar{Y} & \bar{X} \\ -\bar{A} & \bar{R} \end{pmatrix} \begin{pmatrix} R & -X \\ A & Y \end{pmatrix} = \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}. \tag{8}$$

This is called a doubly coprime factorization and has numerous applications in the control area. See Vidyasagar (1985).

In the sequel we will make use of the following technical proposition.

PROPOSITION 2.1. *Let U, V, T be matrix functions in H^∞, with T assumed nonsingular. Then*

1. *Assume U and T are strongly left coprime. Then $V = ZT$ for some $Z \in H^\infty$.*
2. *Assume T is an inner function and U and T are left coprime. Then $V = ZT$ for some $Z \in H^\infty$.*

Proof. 1: Since U, T are strongly left coprime, there exists an H^∞ solution of the Bezout equation

$$TX + UY = I. \tag{9}$$

This implies $X + T^{-1}UY = T^{-1}$ and hence

$$VX + (VT^{-1}U)Y = VT^{-1}.$$

By our assumption $VT^{-1}U \in H^\infty$, and it follows that $VT^{-1}U = Z \in H^\infty$, proving the statement.

2: The function T is assumed inner. We let $P_{H(T)}$ be the orthogonal projection of H^2 onto $H(T)$. Obviously $P_{H(T)} = TP_-T^*$. The left coprimeness of T and U implies that the set $\{P_{H(T)}Uf \mid f \in H^2\}$ is dense in $H(T)$. Now, for $f \in H(T)$ we have

$$\begin{aligned} P_-VT^{-1}P_{H(T)}Uf &= P_-VT^{-1}TP_-T^{-1}Uf \\ &= P_-VP_-T^{-1}Uf = P_-VT^{-1}Uf. \end{aligned}$$

As $VT^{-1}U$ is in H^∞ by assumption, so is $VT^{-1}Uf \in H^2$. By continuity we have $VT^{-1}f \in H^2$ for every $f \in H(T)$. Now, for $f \in TH^2$ we have

$$P_-VT^{-1}Tf = P_-Vf = 0.$$

As $H^2 = H(T) \oplus TH^2$, it follows that $P_{-}VT^{-1}f = 0$ for all $f \in H^2$. This implies that $Z = VT^{-1} \in H^\infty$. ∎

3. SKEW PRIMENESS OF INNER FUNCTIONS

The previous characterization leads directly to the natural question of characterizing those invariant subspaces of $H(P)$ which have complementary subspaces which are also invariant.

In H^∞ there are two natural definition of coprimeness. Of course each notion, in the multivariable context we adopt, comes in a left and a right version. We will say that two H^∞ matrix functions P, R are left coprime if they do not have a nontrivial common inner factor. This is equivalent to the density of $PH^2 + RH^2$ in H^2. We will say that P, R are strongly left coprime if there exists an H^∞ matrix solution to the Bezout equation

$$P(z)X(z) + R(z)Y(z) = I. \tag{10}$$

For an exposition of the connection with the Carleson corona condition we refer to Fuhrmann (1981), where also the following theorem has been proved.

THEOREM 3.1. *Let $P_1H(P_2)$ and $R_1H(R_2)$ be two invariant subspaces of $H(Q)$, with $Q = P_1P_2 = R_1R_2$. Then:*

1. *We have*

$$P_1H(P_2) + R_1H(R_2) = Q_1H(Q_2), \tag{11}$$

where Q_1 is the greatest common left inner factor of P_1, R_1.

2. *We have*

$$P_1H(P_2) \cap R_1H(R_2) = Q_1H(Q_2), \tag{12}$$

where Q_2 is the greatest common right inner factor of P_2, R_2.

3. *We have the direct sum representation*

$$H(Q) = P_1H(P_2) \oplus R_1H(R_2) \tag{13}$$

if and only if P_1, R_1 are strongly left coprime and P_2, R_2 are strongly right coprime.

Let P, $R \in H^\infty$ be inner functions. We extend now Wolovich's (1978) definition of skew primeness to the H^∞ context.

DEFINITION 3.1. We say that the inner functions P and R are *skew prime* if there exist inner functions $\bar{R}$ and $\bar{P}$ such that

1. We have $PR = \bar{R}\bar{P}$.
2. The inner functions P and $\bar{R}$ are strongly left coprime.
3. The inner functions R and $\bar{P}$ are strongly right coprime.

We proceed to give the characterization of skew primeness.

THEOREM 3.2. *Let P and R be inner functions in H^∞. Then the following conditions are equivalent*:

1. *P and R are skew prime.*
2. *$PH(R)$ has a complementary invariant subspace in $H(PR)$.*
3. *There exists an H^∞ solution of the Sylvester equation*

$$X(z)P(z) + R(z)Y(z) = I. \tag{14}$$

Proof. The equivalence of statements 1 and 2 follows from Theorem 3.1. Thus it suffices to prove the equivalence of statements 1 and 3.

$1 \Rightarrow 3$: Assume P and R are skew prime. We can use the equality $PR = \bar{R}\bar{P}$ to define a map $Z: H(R) \to H(\bar{R})$ that intertwines S_R and $S_{\bar{R}}$, i.e. satisfies $ZS_R = S_{\bar{R}}Z$. The map Z is defined by

$$Zf = P_{H(\bar{R})}Pf \qquad \text{for} \quad f \in H(R). \tag{15}$$

The coprimeness conditions included in the definition of skew primeness imply, by Theorem 2.3, that the map Z is invertible. The inverse map is also invertible and by the commutant lifting theorem there exist matrix functions X and $\bar{X}$ in H^∞ such that

$$X\bar{R} = R\bar{X}. \tag{16}$$

The invertibility conditions of Theorem 2.3 imply that X, R are left coprime and $\bar{R}$, $\bar{X}$ are right coprime. The map $Z^{-1}: H(\bar{R}) \to H(R)$ is defined by

$$Z^{-1}f = P_{H(R)}Xf \qquad \text{for} \quad f \in H(\bar{R}). \tag{17}$$

Now, for $f \in H(R)$ we compute

$$f = Z^{-1}Zf = P_{H(R)}XP_{H(\bar{R})}Pf$$
$$= P_{H(R)}XPf, \tag{18}$$

or $P_{H(R)}(I - XP)f = 0$ for $f \in H(R)$. On the other hand, for $f \in RH^2$ we have $f = Rg$ and

$$\begin{aligned}(I - XP)Rg &= Rg - XPRg \\ &= (R - X\bar{R}\bar{P})g = (R - R\bar{X}\bar{P})g \\ &= R(I - \bar{X}\bar{P})g,\end{aligned} \tag{19}$$

i.e.,

$$P_{H(R)}(I - XP)f = 0 \qquad \text{for} \quad f \in H^2. \tag{20}$$

This implies the existence of a matrix function $Y \in H^\infty$ for which $I - XP = RY$ or

$$I = XP + RY, \tag{21}$$

i.e., the Sylvester equation is solvable.

$3 \Rightarrow 1$: Assume there exists an H^∞ solution of the Sylvester equation

$$X(z)P(z) + R(z)Y(z) = I. \tag{22}$$

Clearly, X, R are left coprime and P, Y are right coprime. Given the right coprime factorization YP^{-1}, there exists a left coprime factorization $\bar{P}^{-1}\bar{Y}$ for which

$$YP^{-1} = \bar{P}^{-1}\bar{Y}, \tag{23}$$

or equivalently, for which the intertwining relation

$$\bar{Y}P = \bar{P}Y \tag{24}$$

holds. Note that by the coprimeness conditions we have in particular $\det P = \det \bar{P}$. Now $XP + RY = I$ implies $P^{-1} = X + RYP^{-1} = X + R\bar{P}^{-1}\bar{Y}$, or

$$\begin{aligned} I &= PX + PRYP^{-1} \\ &= PX + PR\bar{P}^{-1}\bar{Y}.\end{aligned}$$

Since $\bar{Y}$, $\bar{P}$ are strongly left coprime, this implies that $\bar{R} = PR\bar{P}^{-1} \in H^{\infty}$. So

$$PR = \bar{R}\bar{P} \tag{25}$$

with $\bar{R}$ necessarily inner. Summing up, we have obtained

$$PX = \bar{R}Y = I \tag{26}$$

and in particular the left coprimeness of P and $\bar{R}$.

We claim now that the matrix

$$\begin{pmatrix} \bar{P} & -\bar{Y} \\ R & X \end{pmatrix}$$

is unimodular, i.e. has a unit determinant. Indeed,

$$\begin{aligned} \begin{pmatrix} \bar{P} & -\bar{Y} \\ R & X \end{pmatrix} &= \begin{pmatrix} \bar{P} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -\bar{P}^{-1}\bar{Y} \\ R & X \end{pmatrix} \\ &= \begin{pmatrix} \bar{P} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -YP^{-1} \\ R & X \end{pmatrix}. \end{aligned} \tag{27}$$

Now

$$\begin{pmatrix} I & -YP^{-1} \\ R & X \end{pmatrix} \begin{pmatrix} I & -YP^{-1} \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ R & X + RYP^{-1} \end{pmatrix}, \tag{28}$$

so

$$\det \begin{pmatrix} I & -YP^{-1} \\ R & X \end{pmatrix} = \det(X + RYP^{-1}). \tag{29}$$

Therefore we have

$$\begin{aligned} \det \begin{pmatrix} \bar{P} & -\bar{Y} \\ R & X \end{pmatrix} &= \det \bar{P} \det(X + RYP^{-1}) \\ &= \det(X + RYP^{-1}) \det P \\ &= \det(XP + RY) = \det I = 1. \end{aligned} \tag{30}$$

Clearly the matrix

$$\begin{pmatrix} I & -YP^{-1} \\ R & X \end{pmatrix}$$

has a unimodular inverse. In view of the equations $\overline{Y}P = \overline{P}Y$ and $XP + RY = I$, there exist H^∞ functions $\overline{X}$, B such that

$$\begin{pmatrix} \overline{P} & -\overline{Y} \\ R & X \end{pmatrix}\begin{pmatrix} \overline{X} & Y \\ -B & Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{31}$$

and hence also

$$\begin{pmatrix} \overline{X} & Y \\ -B & Q \end{pmatrix}\begin{pmatrix} \overline{P} & -\overline{Y} \\ R & X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{32}$$

Now, comparing $-B\overline{P} + PR = 0$ with $PR = \overline{R}\overline{P}$, we conclude that $B = \overline{R}$. So we get the doubly comprime factorization

$$\begin{pmatrix} \overline{X} & Y \\ -\overline{R} & Q \end{pmatrix}\begin{pmatrix} \overline{P} & -\overline{Y} \\ R & X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{33}$$

In particular the Bezout equations $\overline{X}\overline{P} + YR = I$ and $\overline{R}\overline{Y} + QX = I$ are satisfied. This shows the left coprimeness of P, $\overline{R}$ and the right coprimeness of R, $\overline{P}$. This completes the proof. ■

COROLLARY 3.1. *Let P, R be skew prime inner functions. Let $P = P_2 P_1$ and $R = R_2 R_1$ be factorizations with inner factors. Then P_1, R_2 are skew prime.*

Proof. We have, by the skew primeness of P, R, that

$$XP + RY = I. \tag{34}$$

This implies

$$(XP_2)P_1 + R_2(R_1Y) = I. \quad ■ \tag{35}$$

We will need the following lemma.

LEMMA 3.1. *Let P and R be inner functions. Then there exists a nonzero intertwining map $Z: H(P) \to H(R)$ if and only if* det P *and* det R *have a nontrivial common factor.*

Proof. Assume det P and det R are coprime and that $Z: H(P) \to H(R)$ is an intertwining map. Thus there exist A, $B \in H^\infty$ for which $AP = RB$ and

$$Zf = P_{H(R)} Af. \tag{36}$$

From the intertwining relation $AP = RB$ it follows that $R^{-1}AP = B$ and hence, using the equality $pI = \det PI = P \operatorname{adj} P$, that $pR^{-1}A = B \operatorname{adj} P \in H^\infty$. Using Proposition 2.1 and the coprimeness of pI and R, it follows that $R^{-1}A = Q \in H^\infty$, or $A = RQ$. This of course implies that Z, given by (36), is zero.

To prove the converse we will treat first the scalar case. Thus, assume that p, r are inner functions which have a nontrivial greatest common inner divisor t. So $p = tp_1$, $r = tr_1$, and p_1, r_1 are coprime. We consider the following equality:

$$p_1 r = p_1(tr_1) = pr_1.$$

We define a map $Z: H(r) \to H(p)$ by

$$Zf = P_{H(p)} p_1 f \qquad \text{for} \quad f \in H(r).$$

It is easy to show that

$$\ker Z = tH(r_1) \subset H(r),$$

and thus $\ker z = H(r)$ if and only if t is a trivial inner factor, which we assumed is not the case. So the intertwining map Z is nonzero.

To get the general case, let us assume that det P and det R have a nontrivial common inner factor. The map S_P is quasisimilar to $S_{\bar{P}}$, where

$$\bar{P} = \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_m \end{pmatrix},$$

where p_i are the invariant inner factors; see Fuhrmann (1981) for the details. We assume the factors are ordered so that p_i divides p_{i-1}. Similarly let S_R

be quasisimilar to $S_{\bar{R}}$, and $\bar{R}$ similarly defined. Now by our assumption it follows that necessarily p_1 and r_1 have a nontrivial common inner factor. This shows that there exists a nonzero map Z_1 intertwining S_{p1} and S_{r1}. This is easily lifted to a nonzero map that intertwines $S_{\bar{P}}$ and $S_{\bar{R}}$. This immediately implies, by the transitivity of quasisimilarity, that there exists also a nonzero map that intertwines S_P and S_R, and we are done. ∎

We proceed next to discuss the question of the uniqueness of a complementary invariant subspace, assuming it exists.

THEOREM 3.3. *Let P and R be skew prime inner functions. The $PH(R)$ has a unique complementary invariant subspace in $H(PR)$ if and only if* det P *and* det R *are coprime, i.e. have no common nontrivial inner factor.*

Proof. Assume det P and det R to be coprime. Let $PR = \bar{R}\bar{P} = \bar{R}_1\bar{P}_1$ be two factorizations corresponding to the complementary invariant subspaces $\bar{R}H(\bar{P})$ and $\bar{R}_1 H(\bar{P}_1)$, with the corresponding coprimeness condition satisfied. It follows that

$$\det R = \det \bar{R} = \det \bar{R}_1. \tag{37}$$

Consider $Q = \bar{R}^{-1}\bar{R}_1 = \bar{P}\bar{P}_1^{-1}$. By the coprimeness assumption on the determinants we necessarily have $Q \in H^\infty$. Moreover, Q is an inner function. Considering the determinantal equality (37), it follows that Q is a constant unitary matrix.

Conversely, suppose there exists a unique complementary subspace. So $PR = \bar{R}\bar{P}$ and the coprimeness conditions hold. If det P, det R are not coprime, then there exist, by Theorem 2.2, $A, B \in H^\infty$ such that $AP = RB$ and the map $Z: H(P) \to H(R)$ defined by

$$Zf = P_{H(R)} Af \tag{38}$$

is nonzero.

Applying Theorem 3.2, there exists Y, $Y \in H^\infty$ solving the Sylvester equation

$$XP + RY = I. \tag{39}$$

Combining this with the equality $AP = RB$, we get that

$$(X + A)P + R(Y - B) = I \tag{40}$$

is another solution to the Sylvester equation. This leads to

$$(Y - B)P^{-1} = YP^{-1} - BP^{-1} = \bar{P}^{-1}\bar{Y} - R^{-1}A. \tag{41}$$

On the other hand $(Y - B)P^{-1} = \bar{P}_1^{-1}\bar{Y}_1$ with $\bar{P}_1$, $\bar{Y}_1$ left coprime. That leads to a factorization $\bar{R}_1\bar{P}_1 = PR = \bar{R}\bar{P}$. Moreover P, $\bar{R}_1$ are left coprime and R, $\bar{P}_1$ right coprime. By assumption $\bar{P}_1 = \bar{P}$ up to a constant unitary left factor. Without loss of generality we may assume $\bar{P}_1 = \bar{P}$. From the equality $\bar{P}^{-1}Y_1 = \bar{P}^{-1}Y - R^{-1}A$ we get

$$Y - Y_1 = \bar{P}R^{-1}A. \tag{42}$$

Assuming $A = SA_1$ and $R = SR_1$ (i.e., S is the greatest common inner left factor of A, R), we get $\bar{P}R^{-1}A = \bar{P}R_1^{-1}A_1 \in H^\infty$. It follows that $\bar{P} = QR_1$. But then R_1 is a common right inner factor of $\bar{P}$, R, hence necessarily trivial. This implies $A = RD$, and hence the map Z defined by (38) is the zero map, contradicting our assumption. ■

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